SIMPLE COMPLETE BOOLEAN ALGEBRAS†

BY

THOMAS J. JECH

ABSTRACT

If $V = L$, and κ is an uncountable regular non weakly compact cardinal, then there exists a simple complete Boolean algebra of cardinality κ.

1. Introduction

A complete Boolean algebra B is *simple* if it is atomless and if it has no proper atomless complete subalgebra. In this paper we concern ourselves with the question whether simple complete Boolean algebras exist, and if they do, in what sizes.

The problem of existence of simple complete Boolean algebras appears for the first time in 1971 in an article by McAloon [8]. Previously, in 1966, McAloon constructed a rigid complete Boolean algebra, that is, an algebra whose only automorphism is the identity (cf. [7]). It can be readily seen that a simple complete Boolean algebra is rigid.

(If π is a nontrivial automorphism of B, find $x \in B$ such that $\pi x \cdot x = 0$; let C be the subalgebra generated by $D \cup W$, where D denotes the set of all $d \in B$ such that $d \cdot (x + \pi x) = 0$ and W is the set of all elements of the form $z + \pi z$, with $z \leqq x$. Then C is a proper atomless complete subalgebra of B.)

As a matter of fact, it is proved in [8] that an atomless complete Boolean algebra is simple if and only if it is rigid and minimal. Consequently, McAloon observes that the algebra constructed by Jensen in 1968 in L is simple (cf. [5]).

(The algebra used by Jensen to obtain a model with a nonconstructible Δ^1_3 real of minimal L-degree.)

---

† Research supported by National Science Foundation Grant PO-34191-X00.

Received July 29, 1973

I remark in [4] that the construction I gave in 1967 [2] yields a simple complete Boolean algebra (in a model). In the present paper we carry out the construction of a simple complete Boolean algebra in detail; rather than forcing we use Jensen's principle $\diamond$. Thus we obtain another example of a simple complete Boolean algebra in L. The construction is a refinement of the construction of a Suslin tree. By Jensen [6], the construction of a κ-Suslin tree in L works for all regular non weakly compact cardinals; similarly, one can generalize the construction of simple complete Boolean algebras.

Finally, it turns out that the restriction to regular non weakly compact cardinals is necessary; that makes Theorem 3 in L best possible.

We will prove the following theorems.

THEOREM 1. *If κ is a weakly compact cardinal then there is no simple complete Boolean algebra of cardinality κ.*

THEOREM 2. *Assume the generalized continuum hypothesis. If κ is a singular cardinal then there is no simple complete Boolean algebra of cardinality κ.*

THEOREM 3. *Assume the axiom of constructibility. If κ is an uncountable regular cardinal and κ is not weakly compact, then there exists a simple complete Boolean algebra of cardinality κ.*

Therefore, in the universe L of constructible sets. we have simple complete Boolean algebras of arbitrarily large sizes. This does not say anything about the existence of simple complete Boolean algebras in the universe of all sets. It may as well be the case that there are models where simple complete Boolean algebras do not exist; however, we do not wish to conjecture either way.

It seems to us that one should first look into the problem of existence of simple complete Boolean algebras which satisfy the countable chain condition. (Note that one possible tool in this case is Martin's axiom which as yet has no generalization to larger cardinals.) One may find the following observation helpful.

PROPOSITION. *If B is a simple complete Boolean algebra satisfying the countable chain condition then either there exists a Suslin tree or B is countably generated.*

In Theorem 2, we assume the generalized continuum hypothesis (in fact, the assumption is weaker). It would be interesting to have a model with a simple complete Boolean algebra of singular cardinality, for example, $2^{\aleph_0} = \aleph_{\omega_1}$.

As mentioned earlier, being simple coincides with being rigid and minimal.

Since complete Boolean algebras were predestined to serve as a tool in construction of generic models, it is only natural to look at the properties of the generic sets obtained when using simple complete Boolean algebras. Using well-known facts from the theory of Boolean-valued models, we obtain, for example,

COROLLARY (to the proof of Theorem 3). *It is consistent that there exists a set* $X \subseteq \omega_1$ *such that*

(i) X *is nonconstructible*;

(ii) $X \cap \alpha$ *is constructible for each* $\alpha < \omega_1$;

(iii) X *is of minimal L-degree, that is, for every* $Y \in L[X]$, *either* $Y \in L$ *or* $X \in L[Y]$;

(iv) X *is ordinal definable in* $L[X]$.

2. Preliminaries

We use the standard set theoretical notation and terminology. Ordinal numbers are denoted by letters $\alpha, \beta, \gamma, \cdots$ and as customary, an ordinal number coincides with the set of all smaller ordinals. Infinite cardinals are identified with initial ordinals and are denoted by $\kappa, \lambda, \cdots$. The symbol $|X|$ denotes the cardinality of the set X. Each set has a rank, an ordinal number, and we use V_α to denote the collection of rank less than α.

A cardinal κ is *regular* if it is not the sum of less than κ of smaller cardinals; otherwise κ is *singular*. If κ is a given uncountable regular cardinal, a set $C \subseteq \kappa$ is *closed unbounded* if it is closed in the order topology of ordinals and unbounded in κ. A set $S \subseteq \kappa$ is *stationary* if it intersects every closed unbounded subset of κ.

Weakly compact cardinals are defined in various ways. The most suitable definition for our purpose is the definition involving Π^1_1 indescribability. A sentence σ of the second order logic is Π^1_1 if it is of the form $\forall X \phi(X)$ where X is a second order variable and ϕ has no second order quantifiers. A cardinal κ is weakly compact if and only if for every relation R on V_κ and every Π^1_1 sentence σ,

$$\text{if } (V_\kappa, \in, R) \models \sigma \text{ then } \exists \alpha < \kappa (V_\alpha, \in, R \restriction V_\alpha) \models \sigma.$$

The concept of weak compactness is stronger than inaccessibility.

The reader is assumed to be familiar with the notion of complete Boolean algebra. We use $+$ and $-$ to denote the Boolean algebra operations, and Σ and Π for the infinitary operations. Every Boolean algebra has the greatest element 1 and the least element 0, and is partially ordered by $\leqq$. It should be noted that the operations are definable in terms of $\leqq$ and vice versa.

A nonempty subset C of a complete Boolean algebra B is a (complete) *subalgebra* if it is closed under the infinitary operations and under $-$. C is *generated* by $A \subseteq C$, if C is the least subalgebra containing A. A set $D \subseteq B$ is *dense* in B if for every $0 \neq b \in B$ there is $0 \neq d \in D$ such that $d \leqq b$. Two elements $b, c \in B$ are *incompatible* (or *disjoint*) if $b \cdot c = 0$. A set $A \subseteq B$ is a partition of B if the elements of A are pairwise incompatible and if $\Sigma\{a : a \in A\} = 1$. A is a partition of $u \in B$ if instead $\Sigma\{a : a \in A\} = u$. A partition A_2 is *finer* than a partition A_1 if every element of A_1 is partitioned by a subset of A_2.

A complete Boolean algebra B satisfies the *countable chain condition* if every partition of B is at most countable. More generally, B satisfies the *κ-chain condition* if every partition of B is of cardinality less than κ.

An element $a \in B$ is an *atom* if $a \neq 0$ and there is no x such that $0 < x < a$. B is *atomless* if it has no atoms.

If $u \neq 0$ and $A \subseteq B$ we say that A *slices* u if there is $a \in A$ such that $a \cdot u \neq 0$ and $-a \cdot u \neq 0$. A slices a set $S \subseteq B$ if A slices every nonzero $u \in S$.

LEMMA 1. *Let B be a complete Boolean algebra and C a subalgebra of B. Then C is atomless if and only if C slices B.*

PROOF. It is obvious that if C slices C that C is atomless. On the other hand, let $u \neq 0$ be an element of B that is not sliced by C. We let $v = \Pi\{c \in C : c \geqq u\}$ and claim that v is an atom of C. If $x \in C$ is such that $0 \leqq x \leqq v$ then either $x \geqq u$ or $x \cdot u = 0$. If $x \geqq u$ then $x = v$ since v is the least $x \in C$ that $x \geqq u$; if $x \cdot u = 0$ then $v - x = v$ and hence $x = 0$.

3. Proof of Theorem 1

Let κ be a weakly compact cardinal and let B be an atomless complete Boolean algebra of cardinality κ. We use the Π^1_1- indescribability of κ to show that B has an atomless complete subalgebra of smaller cardinality.

We may assume that $B = \kappa$ and thus let $(\kappa, +, \cdot, -)$ be an atomless complete Boolean algebra. The algebra satisfies the κ-chain condition (from a partition of cardinality κ one can obtain 2^κ distinct elements). Hence if $X \subseteq \kappa$ is pairwise incompatible then $X \in V_\kappa$.

Let R be the set of all pairs (a, u) such that $a \subseteq \kappa$, $a \in V_\kappa$, and $u = \Sigma\{b : b \in a\}$. The above discussion shows that the following Π^1_1 sentence σ holds in the model

$\langle V_\kappa, \in, +, \cdot, -, R\rangle$ (the lower-case letter denotes first order variables, the capital letters the second order).

$$\left.\begin{array}{l}(\kappa, +, \cdot, -) \text{ is an atomless Boolean algebra}\\ \text{and } \forall a \text{ (if } a \subseteq \kappa \text{ then } \exists u \text{ such that } (a, u) \in R)\end{array}\right\} \text{first order}$$
$$\text{and } \forall X \text{ (if } X \subseteq \kappa \text{ is pairwise incompatible then } \exists x\ (x = X)).$$

By Π^1_1-indescribability there exists $\alpha < \kappa$ such that σ holds in

$$\langle V_\alpha, \in (+, \cdot, -, R) \restriction V_\alpha\rangle.$$

Therefore, $C = (\alpha, +, \cdot, -)$ is an atomless algebra and $|C| < \kappa$. To show that C is a complete subalgebra of B, we show that $\Sigma\{b : b \in X\} \in C$ for each $X \subseteq C$.

We show this by induction on $|X|$. Let $X = \{b_\xi : \xi < \gamma\}$ and let C be closed under Σ of less than γ elements. Define $c_\xi = b_\xi - \Sigma_{\eta<\xi} b_\eta$, for each $\xi < \gamma$. By induction hypothesis, each $c_\xi \in C$, and $\Sigma\{c_\xi : \xi < \gamma\} = \Sigma\{b_\xi : \xi < \gamma\}$; moreover, the c_ξ are incompatible. Let $Y = \{c_\xi : \xi < \gamma\}$. Since σ holds in V_α, we have $Y \in V_\alpha$; using σ again, we obtain $u \in C$ such that $(Y, u) \in R$, therefore $u = \Sigma\{c_\xi : \xi < \gamma\}$. Hence $\Sigma X \in C$.

REMARK. The proof also gives us the following. If κ is weakly compact then there is no complete Boolean algebra of cardinality κ with less than κ generators. Compare this with the construction of Stavi [9] of a countably generated complete Boolean algebra of cardinality κ for every inaccessible non-Mahlo cardinal κ.

4. Proof of Theorem 2

Assume the generalized continuum hypothesis. Let κ be a singular cardinal and let B be an atomless complete Boolean algebra of cardinality κ. We will construct an atomless complete subalgebra C of smaller cardinality.

The algebra B satisfies the κ-chain condition. However, by a theorem of Erdös and Tarski [1], if λ is the least cardinal such that B satisfies the λ-chain condition, then λ is regular; therefore, $\lambda < \kappa$.

First we construct a subset $A \subseteq B$ of cardinality at most λ such that A slices B.

By induction on $\alpha < \lambda$, we construct partitions A_α such that if $\alpha < \beta$ then $\Sigma A_\beta \leqq \Sigma A_\alpha$ and A_β is finer than A_α. Then we let $A = \bigcup_{\alpha<\lambda} A_\alpha$. By the λ-chain condition, $|A_\alpha| < \lambda$ for each α, and so $|A| \leqq \lambda$.

Let $A_0 = \{1\}$. Having constructed A_α, we pick for each $u \in A_\alpha$ two incompatible nonzero elements v and w such that $v + w = u$. The collection $A_{\alpha+1}$ of all such elements is a finer partition than A_α, and $\Sigma A_{\alpha+1} = \Sigma A_\alpha$.

If α is a limit ordinal, we let A_α be the collection of all nonzero products $\Pi \{u_\beta : \beta < \alpha\}$, where $u_\beta \in A_\beta$ for each $\beta < \alpha$. A_α is a finer partition than each A_β and $\Sigma A_\alpha \leqq \Sigma A_\beta$ for each $\beta < \alpha$.

We claim that A slices B. Otherwise, let $u \neq 0$ be an element not sliced by A. For each $\alpha < \lambda$, there exists a unique $u_\alpha \in A_\alpha$ such that $u_\alpha \geqq u$. It is clear from the construction of A that the sequence $\{u_\alpha : \alpha < \lambda\}$ is strictly decreasing. However, that contradicts the λ-chain condition. (Note that this procedure also gives the proof of the Proposition in Section 1.)

Now we let C be the complete subalgebra of B generated by A. Since A slices B, C is atomless, and so it suffices to show that $|C| < \kappa$. Here we use the generalized continuum hypothesis (in fact, all we need now is $2^\lambda = \lambda$). By induction on $\alpha < \lambda$, we define subsets C_α of B as follows:

$C_0 = A$

$C_\alpha = \bigcup_{\beta<\alpha} C_\beta$ if α is a limit ordinal

$C_{\alpha+1}$ = all possible Σ of less than λ elements of C_α and their complements.

We have $|C_\alpha| \leqq \lambda$ for each $\alpha < \lambda$, and using the λ-chain condition, it follows that $C = \bigcup_{\alpha<\lambda} C_\alpha$; therefore $|C| = \lambda < \kappa$.

5. Proof of Theorem 3

The construction is based on the construction of a Suslin tree. Let T be a κ-Suslin tree; if we reverse the partial order of T and embed it as a dense set in a complete Boolean algebra, then the algebra satisfies the κ-chain condition (and assuming the generalized continuum hypothesis, its cardinality is κ).

Jensen has shown in [6] that in L, a κ-Suslin tree exists for every regular uncountable non weakly compact cardinal. We will use Jensen's technique to construct a κ-Suslin tree with the additional property that the resulting complete Boolean algebra is simple.

First we recall some terminology concerning trees (for more details, see, for example, [3]). A *tree* is a partially ordered set $(T, <)$ such that for every $x \in T$, the set of all predecessors $\{y \in T : y < x\}$ is well ordered; the order type of this set is the *order* of x. The αth level U_α of the tree consists of all $x \in T$ of order α.

T_α is the tree on $\bigcup_{\beta<\alpha} U_\beta$. A *branch* of a tree is a linearly ordered subset containing all predecessors of all its elements; an *antichain* is a set of pairwise incomparable elements.

Let κ be a regular uncountable cardinal. A *normal κ-tree* is a tree with exactly κ levels such that:

(i) every point has successors on all higher levers and at least two immediate successors;

(ii) every branch of limit length has at most one immediate successor;

(iii) every level has cardinality less than κ.

A normal κ-tree is a *κ-Suslin tree* if, moreover, every antichain has cardinality less than κ.

Let $(T, <_T)$ be a κ-Suslin tree. Let $<_B$ be the inverse of the ordering $<_T$: $x <_B y$ iff $x >_T y$. The partially ordered set $(T, <_B)$ can be embedded into a unique (up to isomorphism), complete Boolean algebra $(B, <_B)$ such that T is dense in B. Incomparable elements of T are incompatible as elements of B, and since T is a κ-Suslin tree, B satisfies the κ-chain condition.

Let $\alpha < \kappa$. For each $u \subseteq U_\alpha$ (the αth level), let $[u] = \sum \{x : x \in u\}$. Since T is dense in B, we have $[U_\alpha] = 1$; also, $[u] \leqq_B [v]$ iff $u \subseteq v$. Let $B'_\alpha = \{[u] : u \subseteq U_\alpha\}$; B'_α is a complete subalgebra of B and is isomorphic to the set algebra $\mathscr{P}(U_\alpha)$. Let $B_\alpha = \bigcup_{\beta<\alpha} B_\beta$. B_α is described by T_α and is a (not necessarily complete) subalgebra of B. If a is an arbitrary element of B, then, since T is dense and satisfies the κ-chain condition, there exists a set $X \subset T$, $|X| < \kappa$, such that $a = \sum X$. Then we can find $\alpha < \kappa$ and $u \subseteq U_\alpha$ such that $a = \sum u$; therefore $a \in B'_\alpha$. Consequently, we have

$$B = \bigcup_{\alpha<\kappa} B_\alpha.$$

Now, let C be a complete subalgebra of B. First a trivial but useful remark.

LEMMA 2. *Assume that for each $x \in T$ there exists $y >_T x$ such that $y \in C$. Then $C = B$.*

PROOF. If $a \in B$ then $a = \sum \{x \in T : x \leqq_B a\}$. By the assumption, $a = \sum \{y \in T : y \leqq_B a$ and $y \in C\}$. Therefore $a \in C$.

Let $C'_\alpha = C \bigcap B_\alpha$ and $C_\alpha = C \bigcap B_\alpha$. Since $B = \bigcup_{\alpha<\kappa} B_\alpha$, we have $C = \bigcup_{\alpha<\kappa} C_\alpha$.

Each C'_α is a complete subalgebra of B'_α, hence $\{u \subseteq U_\alpha: [u] \in C\}$ is a complete subalgebra of the set algebra $\mathscr{P}(U_\alpha)$. The collection of minimal $u \subseteq U_\alpha$ such that $[u] \in C$ is a partition of U_α and determines C'_α. Therefore, C_α corresponds to a partition E_α of T_α such that each $e \in E_\alpha$ is a subset of some U_β, and E_α determines C_α.

We say that E_α *slices* T_α if for each $x \in T_\alpha$ there are $y, z \in T_\alpha$ of the same order, both $y >_T x$ and $z >_T x$, and y and z belong to different members of E_α. Note that this exactly means that C_α slices B_α.

LEMMA 3. *If C is an atomless, complete subalgebra of B then the set of all $\alpha < \kappa$ such that E_α slices T_α is closed unbounded.*

PROOF. It is easy to verify that it is closed. To show that it is unbounded, let $\gamma < \kappa$; we find $\alpha > \gamma$ such that E_α slices T_α. For each $x \in U_\gamma$ there is $c \in C$ such that c slices x; therefore there is $\alpha_0 > \gamma$ such that C_{α_0} slices B'_γ. Similarly, there is $\alpha_1 > \alpha_0$ such that C_{α_1} slices B'_{α_0}, and so forth. If we let $\alpha = \lim \alpha_n$, then C_α slices B_α.

Now we say that a partition E of T_α is *good* if every $e \in E$ is a subset of some U_β and if E slices T_α.

We will use combinatorial properties of the constructible universe L to construct a κ-Suslin tree T such that B is a simple complete Boolean algebra, for each regular uncountable nonweakly compact κ.

We present the construction in detail for $\kappa = \omega_1$. In the general case, one has to assume additional combinatorial properties, as in Jensen's proof [6]. We leave it to the reader to verify that the subsequent construction that makes B simple, works also in the general case.

In case $\kappa = \omega_1$, the only assumption we need is the Jensen principle $\diamond$: There is a sequence $\langle S_\alpha : \alpha < \omega_1 \rangle$ such that for each $S \subseteq \omega_1$ the set

$$\{\alpha < \omega_1 : S \cap \alpha = S_\alpha\} \text{ is stationary.}$$

Jensen's construction of a Suslin tree using Jensen's principle can be found in [6]. We will construct a Suslin tree T on ordinals less than ω_1 in such a way that the algebra B is simple.

By induction on α, we construct U_α. If α is a successor, $\alpha = \beta + 1$, then we adjoin to each $x \in U_\alpha$ at least two immediate successors.

If α is a limit ordinal and S_α is a maximal antichain in T_α then we construct U_α so as to nip this antichain in the bud. This will suffice to make T a Suslin tree.

However, we want B to be simple. This is handled as follows. Let α be a limit ordinal such that S_α *is* a pair (x, E) where $x \in T_\alpha$ and E is a good partition of T_α. (We can certainly invent some coding device to give a precise meaning to the word *is*.) We construct U_α as follows. For each $z \in T_\alpha$ there is a branch b_z through z of length α. We appoint immediate successors to some of the b_z; as known, the construction keeps going if for each $y \in T_\alpha$ we appoint a successor to at least one b_z going through y. For every $\beta < \alpha$, let x_β be the βth element of the branch b_x (that is, $b_x = \{x_\beta : \beta < \alpha\}$). We appoint successors only to b_x and to those b_z such that if $z \in U_\beta$ then z is not in the same member of the partition E as x_β. Since E slices T_α, it follows that for each $y \in T_\alpha$ there exists $z >_T y$ such that $z \in U_\beta$ is not in the same member of the partition E as x_β. Hence each $y \in T_\alpha$ will have a successor at level U_α.

Let T be the tree constructed as above. In addition to being Suslin, we show that B is a simple, complete Boolean algebra. Let C be an atomless, complete subalgebra of B and let $x \in T$; we will find $y >_T x$ such that $y \in C$. By Lemma 2 we have $C = B$.

By the introductory remarks, C corresponds to a partition of T, and by Lemma 3, for a closed, unbounded set of α, the partition is a good partition of T_α. Using $\diamond$, we find a limit ordinal α such that $S_\alpha = (x, E_\alpha)$ where E_α is the partition corresponding to C_α and is good.

While constructing U_α, we made sure that b_x is appointed a successor, and only certain b_z's are. Let $y \in U_\alpha$ be the successor of b_x. For each $\beta < \alpha$, let $u_\beta \in E$ be such that $x_\beta \in u_\beta$. We recall that $[u_\beta] \in C$ for each $\beta < \alpha$. Thus we will be done if we show that $y = \Pi \{[u_\beta] : \beta < \alpha\}$ and therefore $y \in C$.

Clearly, $y \leqq_B [u_\beta]$ for each $\beta < \alpha$. To show that $y \geqq_B$ the product, we assume otherwise and then there exists $t \in T$ incomparable with y and $t \leqq_B [u_\beta]$ for each $\beta < \alpha$. This is possible only if there is a branch of length α other than b_x which goes through each u_β and is appointed a successor. However, we constructed U_α such that there is no such branch.

6. Proof of the Corollary

We sketch the proof of the Corollary. Let B be the simple complete Boolean algebra of cardinality $\aleph_1$ that we constructed in L. Let G be an L-generic ultrafilter on B. Since B is atomless, G is not constructible. It is known that, when using a Suslin tree in forcing, then no new countable sets are added; hence every countable subset of G is constructible.

To show that G has a minimal L-degree, let $X \in L[G]$ be nonconstructible. Let

$\underset{\sim}{X}$ be a Boolean-valued name for X and let C be the complete subalgebra of B generated by the values occurring in $\underset{\sim}{X}$. Let $u = \{a \in C: a$ is an atom of $C\}$, $v = -u$. Since $X \notin L$ we have $v \in G$ and $G \bigcup C \in L[X]$. Let D be the complete subalgebra generated by $\{c \in C: c \leqq v\} \cup \{b \in B: b \leqq u\}$. D is an atomless complete subalgebra of B and is a proper subalgebra unless $G \in L[X]$.

To see that G is ordinal-definable in $L[G]$, we refer to an unpublished theorem of Vopěnka saying that if B is a rigid complete Boolean algebra and G_1, G_2 generic ultrafilters on B such that $M[G_1] = M[G_2]$ then $G_1 = G_2$. Therefore in our case, G is definable in $L[G]$ from $B \in L$, hence ordinal definable.

References

1. P. Erdös and A. Tarski, *On families of mutually exclusive sets*, Ann. of Math. (1943), 315–429.

2. T. Jech, *Non-provability of Suslin's hypothesis*, Comment. Math. Univ. Carolinae **8** (1967), 291–305.

3. T. Jech, *Trees*, J. Symbolic Logic **36** (1971), 1–14.

4. T. Jech, *A propos d'algèbres de Boole rigides et minimales*, C. R. Acad. Sci. Paris Sér. A **374** (1972), 371–372.

5. R. B. Jensen, *Definable sets of minimal degree*, Math. Logic and Set Theory, Proc., Bar-Hillel (ed)., Jerusalem, 1968, pp. 122–128.

6. R. B. Jensen, *The fine structure of the constructible hierarchy*, Ann. Math. Logic **4** (1972), 229–308.

7. K. McAloon, *Consistency results about ordinal definability*, Ann. Math. Logic **2** (1971), 449–467.

8. K. McAloon, *Les algèbres be Boole rigides et minimales*, C. R. Acad. Sci. Paris Sér. A **272** (1971), 89–91.

9. J. Stavi, *Cardinal collapsing with reals*, Preliminary report, Amer. Math. Soc. Notices **19** (1972), 764.

State University of New York
Buffalo, New York, U. S. A.

and

Princeton University
Princeton, New Jersey, U. S. A.

and

The Institute for Advanced Study